HOMOGENIZATION-BASED CONSTITUTIVE MODELS FOR TWO-DIMENSIONAL VISCOPLASTIC POROUS MEDIA WITH EVOLVING MICROSTRUCTURE

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ABSTRACT: In this study, we present a model based on the "second-order" nonlinear homogenization method for estimating the macroscopic response and the evolution of microstructure in two-dimensional viscoplastic porous media. The estimates of the new model are compared with unit cell finite element calculations and the earlier "variational" nonlinear homogenization method. The three methods are compared for uniaxial tension and compression, as well as for simple shear loadings. The "second-order" model is found to improve on the earlier "variational" method by being in good agreement with the finite element results.

Keywords: Homogenization, porous, viscoplastic, evolution, microstructure

1 INTRODUCTION

The present work is concerned with the application of the "second-order" nonlinear homogenization procedure of Ponte Castañeda [1], to generate estimates for the effective behavior of viscoplastic porous media. This method is a generalization of the earlier variational method of Ponte Castañeda [2], which has been rigorously shown to lead to uppers bounds for the stress potentials. The focus in this study is on porous composites consisting of cylindrical voids with initially circular cross-section, that are subjected to finite plane-strain deformations. As a consequence of the applied load, the initially circular shape of the voids is expected to evolve into an elliptical one with certain orientation. In this regard, we propose an approximate constitutive model based on the "second-order" nonlinear homogenization method of Ponte Castañeda [1] to estimate the effective behavior of viscoplastic porous materials consisting of cylindrical voids with elliptical cross-section that are subjected to plane-strain loading conditions.

In this context, we consider a representative volume element Ω of a two-phase porous medium with each phase occupying a sub-domain $\Omega^{(r)}$ $(r = 1, 2)$. The vacuous phase is identified as phase 2, while the non-vacuous phase (i.e., matrix phase) is denoted as phase 1. The local behavior of the matrix phase is characterized by a power-law, incompressible isotropic stress potential, such that the Cauchy stress $\boldsymbol{\sigma}$ and the Eulerian strain-rate $\boldsymbol{D}$ at any point in $\Omega^{(1)}$ are related by

$$\boldsymbol{D} = \frac{\partial U^{(1)}}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}), \quad \text{with} \quad U^{(1)}(\boldsymbol{\sigma}) = \frac{\sigma_o}{n+1}\left(\frac{\sigma_{eq}}{\sigma_o}\right)^{n+1}. \tag{1}$$

Here, the von Mises stress is defined in terms of the deviatoric stress tensor as $\sigma_{eq} =$

$\sqrt{\frac{3}{2}\,\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}'}$, σ_o denotes the flow stress of the matrix phase, and $m = 1/n$ is the strain-rate sensitivity parameter, which takes values between, 0 and 1. Note that the two limiting values $m = 1$ and $m = 0$ correspond to linear and ideally-plastic behaviors, respectively.

The effective behavior of the porous material is defined as the relation between the average stress, $\overline{\boldsymbol{\sigma}} = \langle\boldsymbol{\sigma}\rangle$, and the average strain-rate, $\overline{\boldsymbol{D}} = \langle\boldsymbol{D}\rangle$, which can also be characterized by an effective stress potential $\widetilde{U}$, such that [3]

$$\overline{\boldsymbol{D}} = \frac{\partial\,\widetilde{U}}{\partial\,\overline{\boldsymbol{\sigma}}}(\overline{\boldsymbol{\sigma}}), \quad \widetilde{U}(\overline{\boldsymbol{\sigma}}) = (1-f)\inf_{\sigma\,\epsilon\,\mathcal{S}(\overline{\sigma})}\langle U^{(1)}(\boldsymbol{\sigma})\rangle^{(1)}. \tag{2}$$

Here, $\langle\cdot\rangle$ and $\langle\cdot\rangle^{(1)}$ denote volume averages over the representative volume element Ω and over the matrix phase $\Omega^{(1)}$, respectively, f denotes the volume fraction of the porous phase (i.e., the porosity), and $\mathcal{S}\left(\overline{\boldsymbol{\sigma}}\right) = \{\boldsymbol{\sigma}, \mathrm{div}\boldsymbol{\sigma} = 0 \text{ in } \Omega,\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega^{(2)},\ \langle\boldsymbol{\sigma}\rangle = \overline{\boldsymbol{\sigma}}\}$ is the set of statically admissible stresses. It is noted that the porous material is subjected to plane-strain loading conditions, so that $\overline{D}_{i3} = 0$ with $i = 1, 2, 3$.

1.1 The "Second-order" method

To estimate the effective stress potential $\widetilde{U}$ for the porous medium, the "second-order" method, originally proposed by Ponte Castañeda [1], is described next. The method is based on the construction of a "*linear comparison composite*" (LCC), with the same microstructure as the nonlinear composite, whose constituent phases are identified with appropriate linearizations of the given nonlinear phases resulting from a suitably designed variational principle. This allows the use of any method already available to estimate the effective behavior of *linear* composites to generate corresponding estimates for *nonlinear* composites.

For the class of materials considered in this work, the LCC is a porous material, with a matrix phase characterized by

$$U_L\left(\boldsymbol{\sigma};\check{\boldsymbol{\sigma}},\,\mathbf{M}\right) = U\left(\check{\boldsymbol{\sigma}}\right) + \frac{\partial U}{\partial\boldsymbol{\sigma}}\left(\check{\boldsymbol{\sigma}}\right)\,\cdot\,\left(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}\right) + \frac{1}{2}\left(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}\right)\,\cdot\,\mathbf{M}\left(\boldsymbol{\sigma}-\check{\boldsymbol{\sigma}}\right), \tag{3}$$

where the label '1' (denoting the matrix phase) will be omitted for simplicity in the rest of the section. In this expression, the tensor $\check{\boldsymbol{\sigma}}$ is a uniform reference stress tensor, which has been defined for the case of transversely isotropic porous media [4] in such a way that it reproduces exactly the behavior of a "composite-cylinder assemblage" in the limit of in-plane hydrostatic loadings, and therefore coincides with the hydrostatic limit of Gurson's criterion in the special case of ideal plasticity. In turn, $\mathbf{M}$ is a symmetric, fourth-order, compliance tensor of the form [5]

$$\mathbf{M} = \frac{1}{2\lambda}\,\mathbf{E} + \frac{1}{2\mu}\,\mathbf{F}, \quad \text{with} \quad \mathbf{E} = \frac{3}{2}\frac{\check{\boldsymbol{\sigma}}'\otimes\check{\boldsymbol{\sigma}}'}{\check{\sigma}_{eq}^2}, \quad \mathbf{F} = \mathbf{K}-\mathbf{E}. \tag{4}$$

In this last expression, $\mathbf{K}$ denotes the in-plane components of the standard, fourth-order, isotropic, shear projection tensor, whereas the "prime" denotes the deviatoric part of $\check{\boldsymbol{\sigma}}$. Then, the "second-order" estimate for the effective stress potential of the nonlinear porous material is given by [1]

$$\widetilde{U}_{SOM}\left(\overline{\boldsymbol{\sigma}}\right) = \operatorname*{stat}_{\lambda,\,\mu}\left\{\widetilde{U}_L\left(\overline{\boldsymbol{\sigma}};\check{\boldsymbol{\sigma}},\,\mathbf{M}\right) + \,(1-f)\,V\left(\check{\boldsymbol{\sigma}},\mathbf{M}\right)\right\}, \tag{5}$$

where $\widetilde{U}_L$ is the effective stress potential of the LCC, while the "corrector" function V is defined as

$$V\left(\check{\boldsymbol{\sigma}},\mathbf{M}\right) = \operatorname*{stat}_{\hat{\sigma}}\left[U\left(\hat{\boldsymbol{\sigma}}\right) - U_L\left(\hat{\boldsymbol{\sigma}};\check{\boldsymbol{\sigma}},\,\mathbf{M}\right)\right], \tag{6}$$

where $\hat{\sigma}$ is a uniform stress tensor. In these expressions, the stationary operation (stat) consists in setting the partial derivative of the argument with respect to the variable equal to zero, which yields a set of nonlinear equations for the variables λ, μ and $\hat{\boldsymbol{\sigma}}$, as shown next.

Making use of the symmetry of the tensor $\mathbf{M}$, we can define two components of the tensor $\hat{\boldsymbol{\sigma}}$ that are parallel and perpendicular to the corresponding reference tensor $\check{\boldsymbol{\sigma}}$, respectively, $\hat{\sigma}_{||} = (\frac{3}{2}\hat{\boldsymbol{\sigma}}\cdot\mathbf{E}\,\hat{\boldsymbol{\sigma}})^{1/2}$ and $\hat{\sigma}_{\perp} = (\frac{3}{2}\hat{\boldsymbol{\sigma}}\cdot\mathbf{F}\,\hat{\boldsymbol{\sigma}})^{1/2}$, such that the equivalent part of the tensor $\hat{\boldsymbol{\sigma}}$ reduces to $\hat{\sigma}_{eq} = \sqrt{\hat{\sigma}_{||}^2 + \hat{\sigma}_{\perp}^2}$. Taking into account the two stationarity operations described previously in expressions (5) and (6), the effective stress potential of the nonlinear porous composite can be further simplified to [5]

$$\widetilde{U}_{SOM}(\overline{\boldsymbol{\sigma}}) = (1-f)\left[\frac{\sigma_o}{1+n}\left(\frac{\hat{\sigma}_{eq}}{\sigma_o}\right)^{n+1} - \left(\frac{\check{\sigma}_{eq}}{\sigma_o}\right)^{n}\left(\hat{\sigma}_{||} - \frac{\overline{\sigma}_{eq}}{(1-f)}\right)\right], \quad (7)$$

where the quantities $\hat{\sigma}_{eq}$ and $\hat{\sigma}_{||}$ depend on certain traces of the field fluctuations in the LCC. The corresponding macroscopic stress-strain-rate relation follows from differentiation of (5) or, equivalently, (7) and explicit expressions are given in [6, 7].

1.2 Particulate microstructures and evolution

In the present work, the focus is on "particulate" porous materials containing cylindrical voids with initially circular cross-section aligned with the x_3-axis, which are randomly distributed in the transverse plane $x_1 - x_2$. These materials are subjected to finite plane-strain deformations and, as a consequence, the initially circular cylindrical voids evolve into elliptical ones. For this reason, the present model makes of the Willis estimates [8] to determine the effective behavior of the LCC, which in turn provide estimates for the nonlinear porous material. Then, the relevant internal variables characterizing the state of the microstructure are:

1. the volume fraction of the voids or porosity $f = \mathcal{V}_{voids}/\mathcal{V}_{tot}$, where $\mathcal{V}$ denotes volume,

2. the aspect ratio $w = a_2/a_1$ where $2\,a_i$ with $i = 1, 2$ denote the lengths of the in-plane principal axes of the ellipsoidal voids,

3. the angle ψ, which describes the orientation of of the in-plane principal axes of the ellipsoidal voids relative to the fixed laboratory frame of reference.

In addition, the above-described microstructural variables are expected to evolve during the deformation process. The corresponding evolution laws for these variables were given by Ponte Castañeda and Zaidman [9] and Kailasam and Ponte Castañeda [10] (see also [11]) in the context of the "variational" method [2]. These laws can be easily generalized in the context of the "second-order" method, however, due to lack of space these details will be performed elsewhere.

2 DISCUSSION AND RESULTS

In this section, we discuss the implementation of the "second-order" method (SOM) in the case of porous media consisting of cylindrical voids with initially circular cross-section, which are subjected to plane-strain loading conditions. The SOM estimates are compared with unit cell finite element (FEM) results and the earlier "variational" (VAR) predictions.

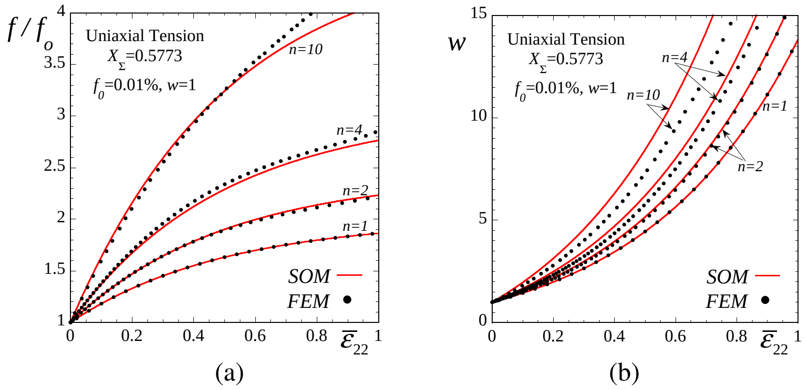


Figure 1: Evolution of the normalized porosity f/f_o and the aspect ratio w for uniaxial tension loading. SOM vs. FEM predictions.

2.1 Aligned loadings

For the calculations performed in this subsection, the initial porosity is considered to be sufficiently small, i.e., $f_o = 0.01\%$, such that the comparison between the *periodic* unit cell and the *random* porous medium, described in the previous section, is meaningful. In turn, traction boundary conditions are applied, so that the only non-zero in-plane, components of the macroscopic stress tensor are

$$\overline{\sigma}_{11} \neq 0, \quad \overline{\sigma}_{22} \neq 0, \quad \overline{\sigma}_{12} = 0, \quad X_\Sigma = \frac{\overline{\sigma}_{11} + \overline{\sigma}_{22}}{\sqrt{3}\,|\overline{\sigma}_{11} - \overline{\sigma}_{22}|}, \tag{8}$$

where X_Σ denotes the stress triaxiality. Provided that the major axis of the voids is aligned with the laboratory frame of reference and the principal loading axes, the only relevant microstructural variables are the porosity f and the in-plane aspect ratio w. In turn, the orientation of the voids remains fixed.

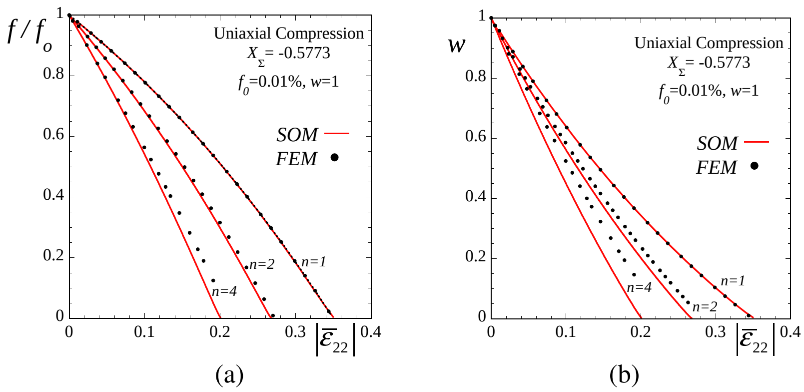
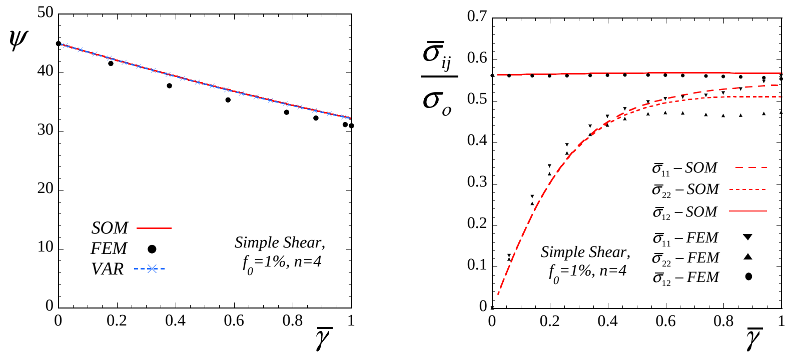


Figure 2: Evolution of the normalized porosity f/f_o and the aspect ratio w for uniaxial compression loading. SOM vs. FEM predictions.

Uniaxial tension loading. Fig. 1 presents results for the evolution of the normalized porosity f/f_o, and the aspect ratio w as a function of the nonlinear exponent $n = 1, 2, 4, 10$ and the total axial strain $\overline{\varepsilon}_{22}$ for uniaxial tension (i.e., $X_\Sigma = 1/\sqrt{3}$). In

Fig. 1a, the predictions of the SOM for the evolution of porosity are in good agreement with the FEM results for all the nonlinearities considered. In turn, looking at Fig. 1b, the corresponding aspect ratio grows substantially for large deformations. In this case, both predictions are in good agreement up to a nonlinearity $n = 4$, whereas for $n = 10$ the SOM overestimates the evolution of w when compared with the FEM. Note that the VAR estimates do not depend on the nonlinearity and thus coincide with the $n = 1$ curve. In this regard, the SOM improves significantly on the VAR method.
Uniaxial compression loading. Fig. 2 shows results for the relevant microstructural and macroscopic variables as a function of the total macroscopic axial strain $|\bar{\varepsilon}_{22}|$ and nonlinearity $n = 1, 2, 4$ for uniaxial compression (i.e., $X_\Sigma = -1/\sqrt{3}$). More specifically, in Fig. 2a, the SOM predictions for the evolution of the normalized porosity f/f_o are in very good agreement with the corresponding results obtained by the FEM. It is noted that it was not possible to have good numerical accuracy with the FEM method for nonlinearities greater than $n = 4$. Looking now at Fig. 2b, the SOM slightly overestimates the evolution of the aspect ratio w when compared with the FEM. However, both methods predict a very sharp change in the aspect ratio, which finally tends to zero as the porosity becomes zero. Similarly to the previous case, the VAR estimates do not depend on the nonlinearity and thus coincide with the $n = 1$ curve.



(a) (b)

Figure 3: Evolution of the angle ψ and the components of the stress tensor in simple-shear loading. SOM.vs.FEM.vs.VAR estimates.

2.2 Simple shear loading

In this subsection, we study the evolution of the orientation of the void when simple shear loading is applied. The material is subjected to total shear strain $2\,\bar{\varepsilon}_{12} = \bar{\gamma}$. For comparison, FEM results are also included. Because of the applied load, the porosity is not evolving. For numerical reasons related to the FEM calculations, the initial porosity has been chosen to be $f = 1\%$.
Fig. 3 presents results for the evolution of the orientation angle ψ and the macroscopic components of the normalized stress tensor $\bar{\boldsymbol{\sigma}}/\sigma_o$ (σ_o denotes the flow stress of the matrix phase) as a function of the applied shear strain $\bar{\gamma}$ for various nonlinearities $n = 1, 2, 4$. Fig. 3a shows the evolution of the orientation angle of a void with initially circular cross-section for a nonlinear exponent $n = 4$. As a consequence of the applied load, the initial orientation of the major axis of the void lies at 45^o. As the deformation progresses, the orientation angle evolves reaching a value of $\sim 32^o$ at shear strain 100%. Both the SOM and the VAR estimates are in good agreement with the FEM predictions. It should be also noted that the evolution of the orientation angle ψ depends very slightly on the nonlinearity. This is the reason that we do

not include graphs for other values of n. In turn, Fig. 3b, shows evolution curves for the normalized macroscopic components of the stress tensor $\overline{\boldsymbol{\sigma}}/\sigma_o$, where the SOM and the FEM are found to be in good agreement. In addition, the shear stress $\overline{\sigma}_{12}$ starts from a finite value, whereas the rest of the components are initially zero. In the sequel, the two components $\overline{\sigma}_{11}$ and $\overline{\sigma}_{22}$ evolve similarly, except at sufficiently large shear strain where they start deviate from each other. On the other hand, the shear stress $\overline{\sigma}_{12}$ remains almost unaffected by the evolution process.

3 CONCLUDING REMARKS

It has been shown that the new model based on the "second-order" method [1] improves significantly on the earlier "variational" [2] estimates by being in much better agreement with the finite element results. Similarly to the "variational" method, the new "second-order" model is able to provide estimates for general in-plane loadings, such as the prediction of the orientation of the voids when the porous medium is subjected to simple shear loading conditions.

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